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Alternating Direction Implicit Schemes for Two-dimensional Generalized Fractional Oldroyd-B Fluids

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Abstract. The two-dimensional Rayleigh-Stokes problem for a generalized fractional Oldroyd-B fluid is considered in the present work. First and second order approximations of the fractional time derivatives are implemented in the developed alternating direction implicit finite difference schemes. Second and compact fourth order approximations are used for the space derivatives. Extensive numerical experiments are performed in order to investigate the stability and accuracy of the proposed algorithms.

INTRODUCTION

Increasing attention has been devoted to the prediction of behavior of viscoelastic non-Newtonian fluids in recent years, due to their broad application in industry (molten plastics, oils and greases, suspensions, emulsions, pulps, *etc.*). The linear constitutive equations containing fractional derivatives in time have proved to be a valuable tool for describing viscoelastic properties, see, *e.g.*, [1] and the references cited there. In particular, a very good fit with experimental data is achieved employing the generalized fractional Oldroyd-B constitutive model, see [2].

In this work the following two-dimensional Rayleigh-Stokes problem for a unidirectional flow of a generalized fractional Oldroyd-B fluid is considered

$$
\begin{aligned}
(1 + aD_t^{\alpha})u_t &= \mu(1 + bD_t^{\beta})\Delta u + f(x, y, t), \quad (x, y) \in (0, 1)^2,\ 0 < t \leq T, && (1)\\
u(x, y, 0) &= u_t(x, y, 0) = 0, \quad (x, y) \in [0, 1]^2, && (2)\\
u(x, y, t) &= v(x, y, t), \quad x = 0 \text{ or } x = 1 \text{ or } y = 0 \text{ or } y = 1;\ \ 0 < t \leq T. && (3)
\end{aligned}
$$

Here $u(x, y, t)$ is the unknown velocity field of the unidirectional flow (in the z-direction), $a, b \geq 0$ and $\mu > 0$ are parameters of the problem, D_t^{α} and D_t^{β} are Riemann-Liouville fractional time derivatives of orders $\alpha \in (0, 1)$ and $\beta \in (0, 1)$, $f(x, y, t)$, $v(x, y, t)$ are given functions. For details on the derivation of Eq. (1) we refer to [3, 4, 5, 6]. Regarding the orders α and β of the fractional derivatives, different assumptions exist in the literature: in [2, 7] the restriction $\alpha \geq \beta$ is imposed, while in other works [5, 6] the restriction $\alpha \leq \beta$ is assumed. Here we consider the whole range $\alpha, \beta \in (0, 1)$. Let us also note that the case $a = 0, b > 0$ corresponds to the generalized fractional second grade model, $b = 0, a > 0$ to the generalized fractional Maxwell model and $a = b = 0$ to Newtonian fluids.

There is a vast number of recent numerical studies of fractional evolution equations, here we will mention only a few of them. Reviews of various methods for approximation of the fractional derivatives can be found in [8, 9, 10]. Based on the first order Grünwald-Letnikov approximation, implicit finite-difference schemes are developed for the fractional diffusion equation in [11, 12, 13] and for the Rayleigh-Stokes problem for a generalized second grade fluid in [14, 15]. The second order Lubich approximation (see [16]) of the fractional derivative is applied in the implicit numerical algorithms for the fractional reaction-diffusion and diffusion-wave equations in [17, 18]. The above mentioned schemes are proved to be unconditionally stable and convergent in the discrete L_2 norm. In [19, 20, 21, 22, 23] implicit schemes are studied for the fractional diffusion/diffusion-wave equations, based on $L1$

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or $L2$ approximation of the fractional time derivative. The accuracy of the $L1$ and $L2$ approximations depends on the order of the fractional derivative. The schemes are unconditionally stable and convergent in the H_1, discrete H_1 or discrete maximum norm. In [12, 13, 17, 18, 21] the second order spatial derivatives are approximated by compact fourth order finite differences.

Alternating direction implicit (ADI) methods use an operator splitting technique to replace the solution of multidimensional problems by solution of independent one-dimensional problems. In this way the advantages of implicitness are retained while solving only banded (usually tridiagonal) systems. ADI schemes are proposed for time-fractional diffusion or diffusion-wave equations in [13, 20, 21, 24].

Theoretical studies of Eq. (1) in one and two dimensions and under different boundary conditions are carried out in [3, 4, 5, 6, 7, 25, 26], where eigenfunction expansions of the solutions are obtained. In [25] estimates for the time-dependent components in these eigenfunction expansions are established, which imply convergence of the series, *i.e.*, problem (1)-(3) admits a unique solution under appropriate smoothness requirements on the data.

Although numerical algorithms for the particular case of problem (1)-(3) with $a = 0$ (corresponding to the generalized second grade fluid) are studied extensively, see, *e.g.*, [14, 15, 27, 28, 29], numerical studies concerning the general case $a \neq 0, b \neq 0$ are still very limited. To the best of the authors' knowledge, the only such work is [6], where a numerical method is developed for the 1D version of problem (1)-(3). In this paper Galerkin finite elements are used in space in combination with $L1$ and $L2$ approximations of the fractional time derivatives, and the unconditional stability and convergence in the H_1-norm is discussed.

Here we present several ADI schemes for the numerical solution of problem (1)-(3), based on the first order Grünwald-Letnikov and the second order Lubich approximations of the fractional time derivatives in combination with second or fourth order finite differences in space. The construction of the corresponding ADI finite-difference schemes is described in the next section. Extensive numerical experiments are presented in the third section in order to numerically investigate the stability and convergence for various values of the parameters α and β. Finally, the advantages and disadvantages of the proposed schemes are discussed.

ADI FINITE-DIFFERENCE SCHEMES

First and Second Order Approximation of Fractional Derivatives

Let us first recall the definitions of the Riemann-Liouville fractional derivative ${}_{RL}D_t^\alpha$ and the Caputo fractional derivative ${}_{C}D_t^\alpha$ of order $\alpha > 0$ (see, *e.g.*, [8])

$$ {}_{RL}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\alpha-1} f(s) \mathrm{d}s, \tag{4} $$

$$ {}_{C}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) \mathrm{d}s. \tag{5} $$

Here n is a positive integer, such that $n-1 < \alpha < n$. The two derivatives are related via the identity (see, *e.g.*, [8])

$$ {}_{RL}D_t^\alpha f(t) = {}_{C}D_t^\alpha f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(0) t^{k-\alpha}}{\Gamma(k+1-\alpha)}. \tag{6} $$

Due to the prescribed homogeneous initial conditions (2) Eq. (6) implies

$$ {}_{RL}D_t^\alpha u = {}_{C}D_t^\alpha u, \quad {}_{RL}D_t^\alpha u_t = {}_{C}D_t^\alpha u_t, \quad \alpha \in (0,1). \tag{7} $$

Therefore the fractional time derivatives in Eq. (1) can be considered in both Riemann-Liouville as well as Caputo sense. Since these two formulations are equivalent, we have not denoted the type of the fractional derivatives (Caputo or Riemann-Liouville) in the equation.

Another implication of the second identity in (7) and the definitions (4) and (5) is that

$$ {}_{RL}D_t^\alpha u_t = {}_{C}D_t^\alpha u_t = {}_{C}D_t^{\alpha+1} u = {}_{RL}D_t^{\alpha+1} u, \quad \alpha \in (0,1). $$

Therefore, Eq. (1) admits the following equivalent formulation

$$u_t + a\, D_t^{\alpha+1} u = \mu(1 + b\, D_t^{\beta})\Delta u + f(x, y, t), \tag{8}$$

which will be used for the construction of the finite difference numerical schemes in the present work.

For the numerical solution of problem (1)-(3) we introduce a uniform discretization in time and in space

$$t_k = k\tau,\ k = 0, 1, \ldots, N_t,\ \tau = T/N_t;\ \ x_i = ih_x,\ i = 0, \ldots, N_x,\ h_x = 1/N_x;\ \ y_j = jh_y, j = 0, \ldots, N_y,\ h_y = 1/N_y,$$

where $N_t + 1, N_x + 1, N_y + 1$ are the numbers of nodes in the corresponding direction.

The Riemann-Liouville fractional derivatives in Eq. (8) are discretized in two ways: applying the first order Grünwald-Letnikov approximation and the second order Lubich approximation defined below.

Finite Grünwald-Letnikov derivative is defined as

$$_{GL}D_t^{\alpha} f(t_k) = \frac{1}{\tau^{\alpha}} \sum_{m=0}^{k} \omega_{1,m}^{\alpha} f(t_{k-m}),\ \ \alpha > 0, \tag{9}$$

where

$$\omega_{1,m}^{\alpha} := (-1)^m \binom{\alpha}{m}. \tag{10}$$

It gives a first order approximation of the Riemann-Liouville derivative for $f \in C^n[0, T]$, $n - 1 \le \alpha < n$ (see [8]):

$$_{RL}D_t^{\alpha} f(t_k) = {}_{GL}D_t^{\alpha} f(t_k) + O(\tau).$$

The coefficients (10) can be evaluated recursively by

$$\omega_{1,0}^{\alpha} = 1,\ \omega_{1,m}^{\alpha} = \left(1 - \frac{\alpha + 1}{m}\right)\omega_{1,m-1}^{\alpha},\ m = 1, 2, \ldots, k.$$

In fact $\omega_{1,m}^{\alpha},\ m = 0, \ldots, k$ are the first $k + 1$ coefficients of the Taylor series expansion of the function

$$W_1^{\alpha}(z) = (1 - z)^{\alpha} = \sum_{m=0}^{\infty} \omega_{1,m}^{\alpha} z^m.$$

The following properties can be easily established:

$$\begin{aligned} &\text{for } 0 < \alpha < 1: \ \ \omega_{1,m}^{\alpha} < 0,\ m \ge 1,\ \ \omega_{1,m}^{\alpha} \ge \omega_{1,m-1}^{\alpha},\ m \ge 2,\ \ \lim_{m\to\infty} \omega_{1,m}^{\alpha} = 0,\\ &\text{for } 1 < \alpha < 2: \ \ \omega_{1,m}^{\alpha} > 0,\ m \ge 2,\ \ \omega_{1,m}^{\alpha} \le \omega_{1,m-1}^{\alpha},\ m \ge 3,\ \ \lim_{m\to\infty} \omega_{1,m}^{\alpha} = 0. \end{aligned}$$

Higher order approximations of the Riemann-Liouville derivative in the form of (9) are given by Ch.Lubich [16]. *Finite Lubich derivatives* satisfy

$$_{RL}D_t^{\alpha} f(t_k) = \frac{1}{\tau^{\alpha}} \sum_{m=0}^{k} \omega_{p,m}^{\alpha} f(t_{k-m}) + O(\tau^p),\ \ \ p = 2, \ldots, 6.$$

The coefficients $\omega_{p,m}^{\alpha}$ are those of the Taylor series expansions of a given generating functions $W_p^{\alpha}(z)$ (see [16])

$$W_p^{\alpha}(z) = \sum_{m=0}^{\infty} \omega_{p,m}^{\alpha} z^m,\ \ p = 2, \ldots, 6.$$

In this paper we consider only the case $p = 2$, in which the generating function is given by

$$W_2^{\alpha}(z) = \left(\frac{3}{2} - 2z + \frac{1}{2}z^2\right)^{\alpha}.$$

Derivation of the coefficients $\omega_{p,m}^{\alpha}$ can be found in [30]. Here we cite the corresponding formulas for $p = 2$

$$\omega_{2,0}^{\alpha} = \left(\frac{3}{2}\right)^{\alpha}, \ \ \omega_{2,1}^{\alpha} = -\frac{4}{3}\alpha\omega_{2,0}^{\alpha},$$

$$\omega_{2,m}^{\alpha} = \frac{2}{3m}\left[-2(\alpha - m + 1)\omega_{2,m-1}^{\alpha} + \frac{1}{2}(2\alpha - m + 2)\omega_{2,m-2}^{\alpha}\right], \ m = 2, 3, \ldots$$

The following properties are satisfied (see [30])

$$\begin{aligned}
&\text{for } 0 < \alpha < 1: \quad \omega_{2,m}^{\alpha} < 0, \ m \geq 4, \ \ \omega_{2,m}^{\alpha} \geq \omega_{2,m-1}^{\alpha}, \ m \geq 5, \ \ \lim_{m\to\infty} \omega_{2,m}^{\alpha} = 0,\\
&\text{for } 1 < \alpha < 2: \quad \omega_{2,m}^{\alpha} > 0, \ m \geq 4, \ \ \omega_{2,m}^{\alpha} \leq \omega_{2,m-1}^{\alpha}, \ m \geq 5, \ \ \lim_{m\to\infty} \omega_{2,m}^{\alpha} = 0.
\end{aligned}$$

When $\alpha = 1$ the well known second order approximation back in time of the first time derivative is obtained:

$$\frac{\partial f}{\partial t}(t_k) = \frac{3f(t_k) - 4f(t_{k-1}) + f(t_{k-2})}{2\tau} + O(\tau^2).$$

Implicit Discretization in Time Using Grünwald-Letnikov Formulas

As was already mentioned in the Introduction, there is a vast number of recent numerical studies on various fractional evolution equations, modeling time-fractional subdiffusion, diffusion-wave, multi-term time-fractional diffusion, or generalized second grade fluids. Many of them use finite difference methods in space and in time, so the proposed here implicit discretization is their natural extension.

Let U_{ij}^{k} be the values of the approximate solution at the nodes $x_i, \ y_j, \ t_k, \ i = 0, \ldots, N_x, \ j = 0, \ldots, N_y, \ k = 0, \ldots, N_t$. Let $U^k = \{U_{ij}^{k}, \ i = 0, \ldots, N_x, \ j = 0, \ldots, N_y\}$. Due to the first initial condition in (2) the solution for $k = 0$ is the trivial one $U^0 = 0$. Due to the second initial condition in (2) the solution for $k = 1$ may be also approximated with the trivial solution.

Using the Grünwald-Letnikov approximation (9) for the fractional derivatives, we can easily write the simplest implicit discretization of Eq. (8) as

$$\frac{U_{ij}^{k+1} - U_{ij}^{k}}{\tau} + \frac{a}{\tau^{\alpha+1}}\sum_{m=0}^{k+1}\omega_{1,m}^{\alpha+1}U_{ij}^{k+1-m} = \mu\Lambda\left(U_{ij}^{k+1} + \frac{b}{\tau^{\beta}}\sum_{m=0}^{k+1}\omega_{1,m}^{\beta}U_{ij}^{k+1-m}\right) + f_{ij}^{k+1}, \tag{11}$$

where $i = 1, \ldots, N_x - 1, \ \ j = 1, \ldots, N_y - 1, \ \ k = 1, \ldots, N_t - 1$, and Λ is the usual second-order discretization of the Laplacian

$$\Lambda = \Lambda_{xx} + \Lambda_{yy}, \ \ \Lambda_{xx}U_{ij} = (U_{i+1,j} - 2U_{ij} + U_{i-1,j})/h_x^2, \ \ \Lambda_{yy}U_{ij} = (U_{i,j+1} - 2U_{ij} + U_{i,j-1})/h_y^2.$$

The Dirichlet boundary conditions (3) are imposed in the usual way

$$U_{ij}^{k+1} = v(x_i, y_j, t_{k+1}), \ \ i = 0, \text{ or } i = N_x, \text{ or } j = 0, \text{ or } j = N_y, \ k = 1, \ldots, N_t - 1.$$

Multiplying (11) by τ and dividing by $1 + a/\tau^{\alpha}$ we get

$$U_{ij}^{k+1} - \tau\mu\frac{1 + b/\tau^{\beta}}{1 + a/\tau^{\alpha}}\Lambda U_{ij}^{k+1} = F_{ij}^{k}(U^k, U^{k-1}, \ldots, U^0, x_i, y_j, t_{k+1}), \tag{12}$$

where

$$F_{ij}^{k}(U^k, U^{k-1}, \ldots, U^0, x_i, y_j, t_{k+1}) := \frac{\tau^{\alpha+1}}{a + \tau^{\alpha}}\left(\frac{U_{ij}^{k}}{\tau} - \frac{a}{\tau^{\alpha+1}}\sum_{m=1}^{k+1}\omega_{1,m}^{\alpha+1}U_{ij}^{k+1-m} + \mu\frac{b}{\tau^{\beta}}\sum_{m=1}^{k+1}\omega_{1,m}^{\beta}\Lambda U_{ij}^{k+1-m} + f_{ij}^{k+1}\right). \tag{13}$$

Let $c := \tau\mu\dfrac{1 + b/\tau^{\beta}}{1 + a/\tau^{\alpha}} = \mu\dfrac{\tau^{\beta} + b}{\tau^{\alpha} + a}\tau^{1+\alpha-\beta}$. Adding the term

$$c^2\Lambda_{xx}\Lambda_{yy}(U_{ij}^{k+1} - U_{ij}^{k}) \tag{14}$$

to the left hand side of Eq. (12) we obtain a product of two one-dimensional operators in front of the unknown solution at level $k+1$

$$(I - c\Lambda_{xx})(I - c\Lambda_{yy})U_{ij}^{k+1} = G_{ij}^{k}, \tag{15}$$

where

$$G_{ij}^{k}(U^k, U^{k-1}, \dots, U^0, x_i, y_j, t_{k+1}) := F_{ij}^{k}(U^k, U^{k-1}, \dots, U^0, x_i, y_j, t_{k+1}) + c^2\Lambda_{xx}\Lambda_{yy}U_{ij}^{k}. \tag{16}$$

Let us introduce $U_{ij}^{*} := (I - c\Lambda_{yy})U_{ij}^{k+1}$, $i = 0, \dots, N_x$, $j = 1, \dots, N_y - 1$. Then, we have to solve for each $j = 1, \dots, N_y - 1$ the following tridiagonal system of linear equations

$$\begin{aligned} (I - c\Lambda_{xx})U_{ij}^{*} &= G_{ij}^{k}, \quad i = 1, \dots, N_{x-1} \\ U_{0,j}^{*} &= (1 - c\Lambda_{yy})v(0, y_j, t_{k+1}) \\ U_{N_x,j}^{*} &= (1 - c\Lambda_{yy})v(x_{N_x}, y_j, t_{k+1}) \end{aligned}$$

and then for each $i = 1, \dots, N_x - 1$ the tridiagonal system of linear equations

$$\begin{aligned} (I - c\Lambda_{yy})U_{ij}^{k+1} &= U_{ij}^{*}, \quad j = 1, \dots, N_{y-1} \\ U_{i,0}^{k+1} &= v(x_i, 0, t_{k+1}) \\ U_{i,N_y}^{k+1} &= v(x_i, y_{N_y}, t_{k+1}). \end{aligned}$$

As all linear systems have strictly diagonally dominant matrices, the Thomas algorithm can be successfully used in order to solve them.

Let us note, that the additional term (14) in (15) corresponds to the following additional term

$$\frac{a + \tau^{\alpha}}{\tau^{\alpha+1}} c^2\Lambda_{xx}\Lambda_{yy}(U_{ij}^{k+1} - U_{ij}^{k})$$

in the discretization (11), *i.e.*, in the procedure described above we have added to (11) a term of order

$$O(c^2\tau/\tau^{\alpha+1}) = O(\tau^{2+2\alpha-2\beta}/\tau^{\alpha}) = O(\tau^{2+\alpha-2\beta}),\ a, b \neq 0.$$

Thus first order approximation of the equation in time is ensured when $\alpha \geq 2\beta - 1$, *i.e.*, if $\beta > 0.5$ we have a restriction for α. We will refer to this discretization as "Method 1."

In [13], where ADI discretization for the 2D time-fractional diffusion equation is considered, the following quantity $\Lambda_{xx}\Lambda_{yy}(U_{ij}^{k+1} - 2U_{ij}^{k} + U_{ij}^{k-1})$ is used instead of $\Lambda_{xx}\Lambda_{yy}(U_{ij}^{k+1} - U_{ij}^{k})$ in order to factorize the 2D discrete operator, when the first order approximation in time can not be achieved in the simplest way.

This means that here we can add the following term to (11)

$$\frac{a + \tau^{\alpha}}{\tau^{\alpha+1}} c^2\Lambda_{xx}\Lambda_{yy}(U_{ij}^{k+1} - 2U_{ij}^{k} + U_{ij}^{k-1}),$$

which is of order $O(\tau^{3+\alpha-2\beta}),\ a, b \neq 0$, if the first time derivative of the solution is smooth. Therefore, in this way the first order approximation is ensured for all $\alpha, \beta \in (0, 1)$. We will call this discretization "Method 2."

We also considered adding the following term to (11)

$$\frac{a + \tau^{\alpha}}{\tau^{\alpha+1}} c^2\Lambda_{xx}\Lambda_{yy}(U_{ij}^{k+1} - 3U_{ij}^{k} + 3U_{ij}^{k-1} - U_{ij}^{k-2}),$$

which is of order $O(\tau^{4+\alpha-2\beta}),\ a, b \neq 0$, when the second time derivative of the solution is smooth. However, in many cases this led to an unstable numerical solution. We will refer to this discretization as "Method 3."

Stability and convergence analysis of the proposed numerical schemes might be performed in a way similar to those in [11, 12, 13, 14, 15, 17, 18]. In these works the Fourier method is applied in order to prove the unconditional stability and convergence in the discrete L_2 norm of some implicit finite difference schemes for the fractional diffusion, diffusion-wave or generalized second grade fluid equation. Here we will present only some numerical experiments in order to investigate numerically stability and convergence of the proposed schemes for numerical solution of problem (1)-(3) for various values of α and β. Detailed theoretical investigation of stability and convergence will be a subject of another work.

Implicit Discretization in Time Using Lubich Formulas

Since the second order Lubich approximation of the fractional time derivatives is less frequently used in the literature than $L1$ and $L2$ approximations, we chose to test numerically the stability and convergence for that case. The theoretical investigation is similar to the case of the Grünwald-Letnikov approximation, and the corresponding results will be published in a future work.

Using the Lubich formulas for the fractional derivatives and the second order approximation back in time for the first time derivative we obtain

$$\frac{3U_{ij}^{k+1} - 4U_{ij}^{k} + U_{ij}^{k-1}}{2\tau} + \frac{a}{\tau^{\alpha+1}} \sum_{m=0}^{k+1} \omega_{2,m}^{\alpha+1} U_{ij}^{k+1-m} = \mu\Lambda \left(U_{ij}^{k+1} + \frac{b}{\tau^{\beta}} \sum_{m=0}^{k+1} \omega_{2,m}^{\beta} U_{ij}^{k+1-m} \right) + f_{ij}^{k+1}.$$

If we multiply by τ and divide by $1.5 + 1.5^{\alpha+1} a/\tau^{\alpha}$ we obtain similar schemes, but for

$$c = c_L := \mu \frac{\tau^{\beta} + b(1.5)^{\beta}}{1.5\tau^{\alpha} + a(1.5)^{\alpha+1}} \tau^{1+\alpha-\beta} = O(\tau^{1+\alpha-\beta}),\ a, b \neq 0.$$

Thus, the additional terms in the discretization (for Method 1, 2 and 3) are of the same order, as in the previous case. The following relations must hold in order to ensure the second order approximation of the equation in time

- for Method 1: $\alpha \geq 2\beta$;
- for Method 2: $\alpha \geq 2\beta - 1$;
- for Method 3: no restriction.

Compact Fourth Order Approximation in Space

We also applied the widely used compact fourth order approximation in space. The main reason was to test the Lubich approximation in some cases, where a very fine grid is needed for the second order approximation in space.

Let us define (see also [12, 13, 17, 18, 21])

$$\Theta_x := I + \frac{h_x^2}{12}\Lambda_{xx},\ \ \Theta_y := I + \frac{h_y^2}{12}\Lambda_{yy},\ \ \Theta := \Theta_y\Lambda_{xx} + \Theta_x\Lambda_{yy}.$$

Then

$$\Theta_x U_{ij} = (U_{i+1,j} + 10U_{ij} + U_{i-1,j})/12,\ \ \Theta_y U_{ij} = (U_{i,j+1} + 10U_{ij} + U_{i,j-1})/12,$$

and

$$\Theta_x \frac{\partial^2 u}{\partial x^2} = \Lambda_{xx} u + O(h_x^4),\ \ \Theta_y \frac{\partial^2 u}{\partial y^2} = \Lambda_{yy} u + O(h_y^4).$$

Multiplying Eq. (8) by $\Theta_x\Theta_y$ and using the Lubich formulas in time we obtain the following finite difference scheme

$$\Theta_x\Theta_y \left(\frac{3U_{ij}^{k+1} - 4U_{ij}^{k} + U_{ij}^{k-1}}{2\tau} + \frac{a}{\tau^{\alpha+1}} \sum_{m=0}^{k+1} \omega_{2,m}^{\alpha+1} U_{ij}^{k+1-m} \right) = \mu\Theta \left(U_{ij}^{k+1} + \frac{b}{\tau^{\beta}} \sum_{m=0}^{k+1} \omega_{2,m}^{\beta} U_{ij}^{k+1-m} \right) + \Theta_x\Theta_y f_{ij}^{k+1}.$$

Using the same coefficient $c = c_L$ and the same additional terms as in the previous case we obtain

$$(I - \tilde{c}_x\Lambda_{xx})(I - \tilde{c}_y\Lambda_{yy})U_{ij}^{k+1} = \tilde{G}_{ij}^{k},$$

where $\tilde{c}_x = c_L - h_x^2/12$, $\tilde{c}_y = c_L - h_y^2/12$,

$$\begin{aligned} \tilde{G}_{ij}^{k}(U^k, U^{k-1}, \ldots, U^0, x_i, y_j, t_{k+1}) \quad &:= \quad \frac{\tau^{\alpha+1}}{1.5^{\alpha+1}a + 1.5\tau^{\alpha}} \left[\Theta_x\Theta_y \left(\frac{4U_{ij}^{k} - U_{ij}^{k-1}}{2\tau} - \frac{a}{\tau^{\alpha+1}} \sum_{m=1}^{k+1} \omega_{2,m}^{\alpha+1} U_{ij}^{k+1-m} \right) \right. \\ &+ \quad \left. \mu \frac{b}{\tau^{\beta}} \sum_{m=1}^{k+1} \omega_{2,m}^{\beta} \Theta U_{ij}^{k+1-m} + \Theta_x\Theta_y f_{ij}^{k+1} \right] + A_{ij}^{k}, \end{aligned}$$

$A_{ij}^k = c_L^2 \Lambda_{xx}\Lambda_{yy} U_{ij}^k$ for Method 1, $A_{ij}^k = c_L^2 \Lambda_{xx}\Lambda_{yy}(2U_{ij}^k - U_{ij}^{k-1})$ for Method 2, and $A_{ij}^k = c_L^2 \Lambda_{xx}\Lambda_{yy}(3U_{ij}^k - 3U_{ij}^{k-1} + U_{ij}^{k-2})$ for Method 3, and

$$U_{ij}^* = (I - \tilde{c}_y \Lambda_{yy}) U_{ij}^{k+1}, \quad j = 1, \dots, N_y - 1, \quad i = 0, \dots, N_x. \tag{17}$$

The coefficient matrices of the resulting linear systems of equations are strictly diagonally dominant, *i.e.*, they can be solved using the Thomas algorithm. The prescribed Dirichlet boundary conditions are imposed in the same way as for the usual second order discretization in space, the only difference is in using $\tilde{c}_y$ in (17) instead of c.

NUMERICAL EXPERIMENTS

Extensive numerical experiments are performed in order to investigate the stability and the accuracy of the solutions for different values of the parameters α and β. The initial data and the right-hand side are chosen to correspond to an exact solution

$$u(x, y, t) = \mathrm{e}^{x+y} t^{\gamma+1},$$

where $\gamma = 3.5$ (see Figure 1). The choice of γ was motivated by two reasons – first, it should not be a natural number, as sometimes a better order of convergence may be obtained for polynomials, and second – the solution should have a sufficient smoothness in order to apply the three methods introduced in the previous section. As several time derivatives of this solution are equal to zero at $t = 0$, we may use the simplest one-sided (in general first order) approximation to the second initial condition $U^1 = U^0$, and if necessary (for Method 3) $U^2 = U^0$.

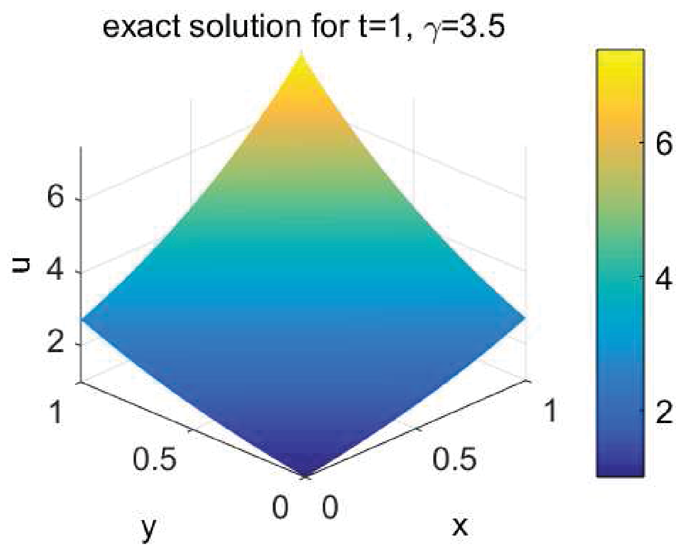

FIGURE 1. The exact solution, used for comparison in the numerical experiments

The order of convergence l is computed using Runge's rule

$$l = \log_2 \frac{\delta(U_{s-1})}{\delta(U_s)},$$

where s is the number of the corresponding grid and

$$\delta(U) := \max\{|u(x_i, y_j, t_k) - U(x_i, y_j, t_k)|, \quad 0 \le i \le N_x, \ 0 \le j \le N_y, \ 0 \le k \le N_t\}$$

is the maximum of the absolute value of the difference between the exact and the numerical solution. In all numerical experiments we will take $a = b = \mu = 1$, $N_x = N_y$, and $T = N_t \tau = 1$.

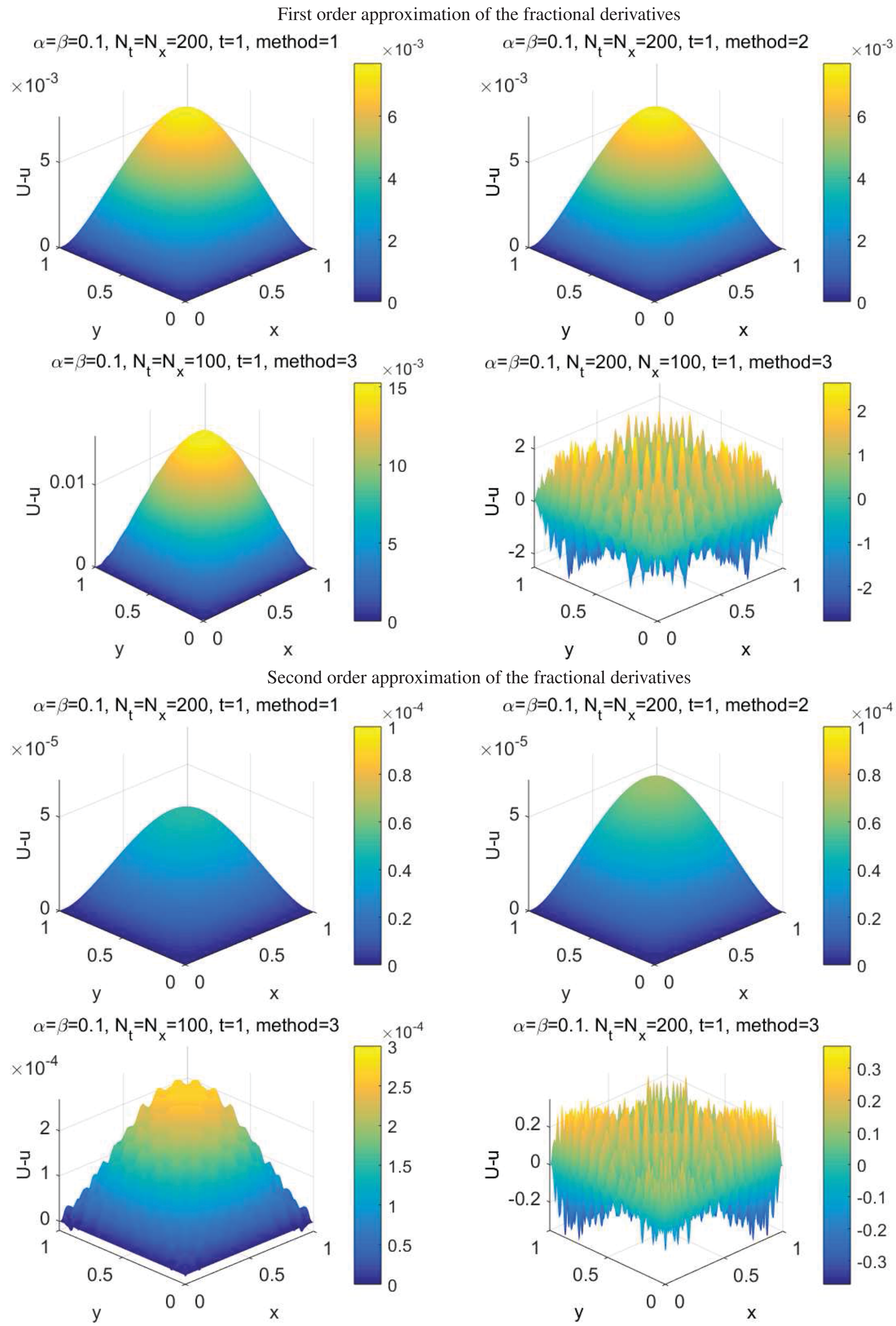

FIGURE 2. The difference between the numerical and the exact solution for $t = 1$

TABLE 1. The error $\delta(U)$ and the order l for $\alpha = 0.1,\ \beta = 0.1,\ 2+\alpha-2\beta = 1.9$

N_t	$N_x = N_y$	Method 1		Method 2	
First order Grünwald-Letnikov approximation in time					
100	100	1.5066e-2		1.5180e-2	
200	100	7.5939e-3	0.9884	7.6216e-3	0.9940
400	100	3.8127e-3	0.9940	3.8196e-3	0.9967
800	100	1.9113e-3	0.9963	1.9131e-3	0.9975
100	100	1.5066e-2		1.5180e-2	
200	200	7.5929e-3	0.9886	7.6206e-3	0.9942
400	400	3.8107e-3	0.9946	3.8176e-3	0.9972
800	800	1.9088e-3	0.9974	1.9106e-3	0.9986
Second order Lubich approximation in time					
100	100	1.9894e-4		2.6217e-4	
200	200	4.9245e-5	2.0143	6.6214e-5	1.9853
400	400	1.2141e-5	2.0201	1.6640e-5	1.9925
800	800	2.9853e-6	2.0239	4.1709e-6	1.9962
6400	5	1.1051e-3		1.1051e-3	
6400	10	2.8527e-4	1.9538	2.8529e-4	1.9537
6400	20	7.1944e-5	1.9874	7.1966e-5	1.9870
6400	40	1.8056e-5	1.9944	1.8077e-5	1.9932
6400	80	4.5508e-6	1.9883	4.5723e-6	1.9832
Compact approximation in space, second order Lubich approximation in time					
100	25	1.9567e-4		2.5861e-4	
200	25	4.8423e-5	2.0147	6.5314e-5	1.9853
400	25	1.1933e-5	2.0207	1.6411e-5	1.9927
800	25	2.9307e-6	2.0256	4.1110e-6	1.9971
100	5	1.9325e-4		2.5651e-4	
400	10	1.1775e-5	4.0367	1.6275e-5	3.9783
1600	20	7.0946e-7	4.0529	1.0210e-6	3.9946
6400	40	4.2537e-8	4.0599	6.4050e-8	3.9946
25600	80	2.6158e-9	4.0234	4.1042e-9	3.9640

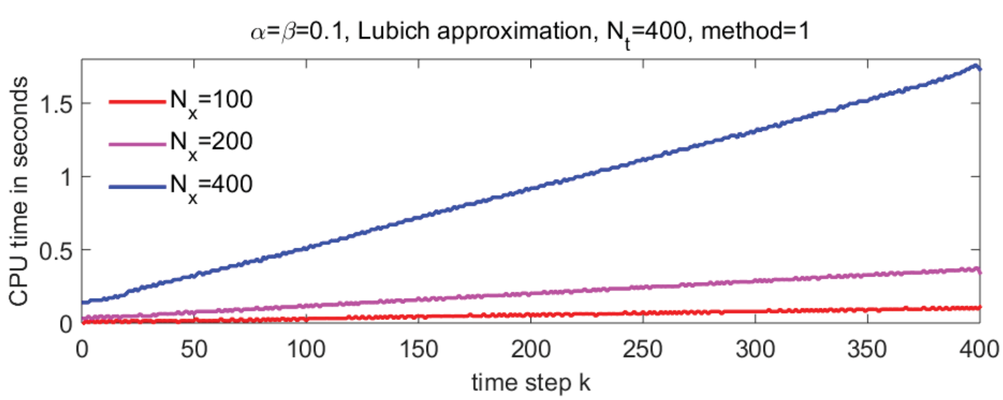

FIGURE 3. The time for computing the solution on different grids

TABLE 2. The error $\delta(U)$ and the order l for $\alpha = 0.5,\ \beta = 0.5,\ 2 + \alpha - 2\beta = 1.5$

N_t	$N_x = N_y$	Method 1		Method 2	
First order Grünwald-Letnikov approximation in time					
100	100	1.6514e-2		1.6871e-2	
200	100	8.3589e-3	0.9823	8.4709e-3	0.9940
400	100	4.2082e-3	0.9901	4.2450e-3	0.9968
800	100	2.1134e-3	0.9936	2.1259e-3	0.9977
100	100	1.6514e-2		1.6871e-2	
200	200	8.3577e-3	0.9825	8.4698e-3	0.9941
400	400	4.2061e-3	0.9906	4.2430e-3	0.9972
800	800	2.1110e-3	0.9946	2.1234e-3	0.9987
Second order Lubich approximation in time					
100	100	1.1377e-4		2.6673e-4	
200	200	2.4027e-5	2.2434	6.7712e-5	1.9780
400	400	1.0677e-5	1.1702	1.7091e-5	1.9852
800	800	5.4765e-6	0.9632	4.2989e-6	1.9912
Compact approximation in space, second order Lubich approximation in time					
100	25	1.1301e-4		2.6322e-4	
200	25	2.3901e-5	2.2413	6.6823e-5	1.9779
400	25	1.0859e-5	1.1382	1.6865e-5	1.9863
800	25	5.5245e-6	0.9750	4.2398e-6	1.9920
1600	25	2.3754e-6	1.2177	1.0615e-6	1.9979
3200	25	9.4697e-7	1.3268	2.6336e-7	2.0110
6400	25	3.6323e-7	1.3824	6.3305e-8	2.0566
12800	25	1.3647e-7	1.4122	1.3968e-8	2.1802

Example 1. The orders of the fractional derivatives are $\alpha = 0.1,\ \beta = 0.1$. In this case $2 + \alpha - 2\beta = 1.9$ and we may expect first order accuracy in time for the Grünwald-Letnikov approximation of the fractional derivatives. For the Lubich approximation we may expect 1.9 order of accuracy in time for Method 1 and second order accuracy in time for Method 2. The results, presented in Table 1, confirm these expectations. Method 3 is not stable in most of the cases and that is why we do not present here the corresponding results.

We may also observe that for the Grünwald-Letnikov approximation it is not strictly necessary to refine the grids used in space, *i.e.*, for every fixed N_t the errors for $N_x = N_y = 100$ and $N_x = N_y = N_t$ are almost the same (see the first and the second groups of data in the table).

On the other hand, when the Lubich approximation is applied in combination with the second order approximation of the Laplacian, it is necessary to refine in space. That is why we also present results with the compact approximation, in which case it is not necessary to use fine grids in space. Even for $N_x = N_y = 25$ almost the same results are achieved as using the usual second order approximation in space for much larger values of N_x and N_y. Of course, we may use the compact scheme only when the solution is sufficiently smooth.

The approximation in space is also investigated, and the results show second order convergence for the usual second order approximation of the Laplacian and fourth order convergence for the compact approximation. In order to demonstrate the fourth order convergence for the compact scheme, either extremely small steps in t should be used for a fixed N_t, or simultaneous refinements in time and space should be applied in an appropriate way. That is why we present results where the steps in t decrease four times and the steps in x and y decrease twice.

Plots of the difference between the numerical and exact solutions of the problem are presented in Figure 2. As can be seen, for Methods 1 and 2 the error is very smooth and symmetric with respect to x and y. Note that the numerical schemes are not symmetric with respect to x and y. But when Method 3 is used, the error highly oscillates and increases on finer grids, which is a strong indication for instability.

The CPU time needed to compute the numerical solution with 400 steps in time and 100, 200 or 400 grid points in each one of the space directions is presented in Figure 3. As can be seen, in the beginning of the computations

TABLE 3. The error $\delta(U)$ and the order l for $\alpha = 0.9,\ \beta = 0.9,\ 2 + \alpha - 2\beta = 1.1$

N_t	$N_x = N_y$	Method 1		Method 2	
First order Grünwald-Letnikov approximation in time					
100	100	1.7633e-2		1.9048e-2	
200	100	8.9931e-3	0.9714	9.5726e-3	0.9927
400	100	4.5484e-3	0.9835	4.7989e-3	0.9962
800	100	2.2915e-3	0.9891	2.4036e-3	0.9975
100	100	1.7633e-2		1.9048e-2	
200	200	8.9916e-3	0.9716	9.5712e-3	0.9929
400	400	4.5462e-3	0.9839	4.7969e-3	0.9966
800	800	2.2889e-3	0.9900	2.4011e-3	0.9984
Second order Lubich approximation in time					
100	100	6.9947e-4		2.4425e-4	
200	200	3.9464e-4	0.8257	6.1964e-5	1.9789
400	400	2.0133e-4	0.9710	1.5686e-5	1.9819
800	800	9.8224e-5	1.0354	3.9630e-6	1.9848
Compact approximation in space, second order Lubich approximation in time					
100	25	7.0215e-4		2.4094e-4	
200	25	3.9517e-4	0.8293	6.1123e-5	1.9789
400	25	2.0140e-4	0.9724	1.5472e-5	1.9821
800	25	9.8211e-5	1.0361	3.9086e-6	1.9849
1600	25	4.6874e-5	1.0671	9.8432e-7	1.9895
3200	25	2.2130e-5	1.0828	2.4617e-7	1.9995
6400	25	1.0386e-5	1.0914	6.5094e-8	1.9191

TABLE 4. The error $\delta(U)$ and the order l for $\alpha = 0.9,\ \beta = 0.1,\ 2 + \alpha - 2\beta = 2.7$

N_t	$N_x = N_y$	Method 1		Method 2	
First order Grünwald-Letnikov approximation in time					
100	100	3.9699e-2		3.9704e-2	
200	100	1.9948e-2	0.9929	1.9949e-2	0.9930
400	100	9.9992e-3	0.9964	9.9993e-3	0.9964
800	100	5.0067e-3	0.9980	5.0067e-3	0.9980
100	100	3.9699e-2		3.9704e-2	
200	200	1.9948e-2	0.9929	1.9949e-2	0.9930
400	400	9.9981e-3	0.9965	9.9981e-3	0.9966
800	800	5.0050e-3	0.9983	5.0050e-3	0.9983
Second order Lubich approximation in time					
100	100	4.8860e-4		4.9018e-4	
200	200	1.2267e-4	1.9938	1.2291e-4	1.9957
400	400	3.0738e-5	1.9968	3.0773e-5	1.9979
800	800	7.6938e-6	1.9983	7.6991e-6	1.9989
Compact approximation in space, second order Lubich approximation in time					
100	25	4.8529e-4		4.8687e-4	
200	25	1.2183e-4	1.9940	1.2207e-4	1.9958
400	25	3.0523e-5	1.9969	3.0559e-5	1.9980
800	25	7.6379e-6	1.9986	7.6432e-6	1.9993

TABLE 5. The error $\delta(U)$ and the order l for $\alpha = 0.1,\ \beta = 0.9,\ 2+\alpha-2\beta = 0.3$

N_t	$N_x = N_y$	Method 1		Method 2		Method 3	
First order Grünwald-Letnikov approximation in time							
100	200	1.6180e-2		1.4388e-3		2.6537e-3	
200	200	1.4920e-2	0.1170	8.3133e-4	0.7914	1.3414e-3	0.9843
400	200	1.3455e-2	0.1491	4.6130e-4	0.8497	6.7413e-4	0.9926
800	200	1.1942e-2	0.1721	2.4956e-4	0.8863	3.3812e-4	0.9955
1600	200	1.0468e-2	0.1901	1.3281e-4	0.9100	1.6961e-4	0.9953
3200	200	9.0875e-3	0.2040	6.9969e-5	0.9246	8.5240e-5	0.9926
6400	200	7.8248e-3	0.2158	3.6700e-5	0.9309	4.3130e-5	0.9828
100	100	1.6178e-2		1.4411e-3		2.6559e-3	
200	200	1.4920e-2	0.1168	8.3133e-4	0.7937	1.3414e-3	0.9855
400	400	1.3456e-2	0.1490	4.6070e-4	0.8516	6.7354e-4	0.9939
800	800	1.1942e-2	0.1722	2.4882e-4	0.8887	3.3733e-4	0.9976
Second order Lubich approximation in time							
100	100	2.0857e-2		1.5429e-3		3.6779e-6	
200	200	1.8862e-2	0.1450	6.6520e-4	1.2138	2.2757e-6	0.6926
400	400	1.6888e-2	0.1595	2.8149e-4	1.2407	1.0317e-6	1.1413
800	800	1.4984e-2	0.1726	1.1800e-4	1.2543	3.5302e-7	1.5472
200	25	1.8817e-2		6.1245e-4		5.3154e-5	
400	50	1.6876e-2	0.1571	2.6870e-4	1.1886	1.3806e-5	1.9449
800	100	1.4980e-2	0.1719	1.1481e-4	1.2267	3.5509e-6	1.9590
1600	200	1.3181e-2	0.1846	4.8419e-5	1.2456	9.0821e-7	1.9671
Compact approximation in space, second order Lubich approximation in time							
100	25	2.0859e-2		1.5421e-3		3.3490e-6	
200	25	1.8863e-2	0.1451	6.6429e-4	1.2150	1.7695e-6	0.9204
400	25	1.6888e-2	0.1596	2.8099e-4	1.2413	8.2782e-7	1.0960
800	25	1.4983e-2	0.1727	1.1777e-4	1.2545	3.0164e-7	1.4565
1600	25	1.3181e-2	0.1849	4.9114e-5	1.2618	9.2731e-8	1.7017
3200	25	1.1502e-2	0.1966	2.0423e-5	1.2659	2.4340e-8	1.9297
6400	25	9.9635e-3	0.2072	8.4768e-6	1.2686	4.3421e-9	2.4869
12800	25	8.5713e-3	0.2171	3.5146e-6	1.2702	3.8719e-8	unstable
25600	25	7.3276e-3	0.2262	1.4568e-6	1.2706		

($k = 1$) the CPU time is much smaller than in the end of the computations ($k = 400$). As only the time for computing the sums in the approximate formulas for the fractional derivatives (see (13) and also (15), (16)) depends linearly on k, we may conclude that the computation of the fractional derivatives is the most time consuming part of the numerical method. Note that the time for solving the resulting linear systems of equations on each time step does not depend on k, *i.e.*, it may be estimated by the CPU time at $k = 1$. As a result, if an explicit method will be used, it will have to compute the same sums for the fractional derivatives, therefore it can not be much faster. However there will be a strong condition for its stability. In fact when we used an explicit method approximately about 50000 time steps had to be used in order to ensure stability for $N_x = N_y = 100$.

Let us also note that in order to compute the fractional derivatives, we have to store the solution on all time steps. If a grid with $N_x = N_y = 1000$ points is used, and the number of time steps is also $N_t = 1000$, we will need to use about 8×10^9 bytes, *i.e.*, about 7.5 GB memory. In order to avoid the multiple calculation of $\Lambda U^{k+1-m},\ m = 1, \ldots, k+1$ (or ΘU^{k+1-m}, $\Theta_x \Theta_y U^{k+1-m}$ for the compact scheme) we may also store such quantities on the already computed time levels, but then we will need to use additional GBs of memory. Thus, it may also be important to have a balance between faster computations and the available memory. A possible remedy is to use the short memory principle (see [8]), *i.e.*, to truncate the "tails" of the fractional derivatives.

TABLE 6. The error $\delta(U)$ and the order l for $\alpha = 0.1,\ \beta = 0.5,\ 2 + \alpha - 2\beta = 1.1$

N_t	$N_x = N_y$	Method 1		Method 2	
First order Grünwald-Letnikov approximation in time					
100	100	7.9286e-3		9.3329e-3	
200	100	4.0811e-3	0.9581	4.6970e-3	0.9906
400	100	2.0789e-3	0.9731	2.3568e-3	0.9949
800	100	1.0537e-3	0.9804	1.1815e-3	0.9962
100	100	7.9286e-3		9.3329e-3	
200	200	4.0791e-3	0.9588	4.6951e-3	0.9912
400	400	2.0763e-3	0.9742	2.3543e-3	0.9959
800	800	1.0509e-3	0.9824	1.1787e-3	0.9981
Second order Lubich approximation in time					
100	100	1.0077e-3		1.2271e-4	
200	200	5.0801e-4	0.9881	3.1768e-5	1.9496
400	400	2.4698e-4	1.0405	8.1576e-6	1.9614
800	800	1.1820e-4	1.0632	2.0836e-6	1.9691
Compact approximation in space, second order Lubich approximation in time					
100	25	1.0090e-3		1.1936e-4	
200	25	5.0794e-4	0.9902	3.0920e-5	1.9487
400	25	2.4678e-4	1.0414	7.9417e-6	1.9610
800	25	1.1807e-4	1.0636	2.0266e-6	1.9704
1600	25	5.6127e-5	1.0729	5.1307e-7	1.9818
3200	25	2.6616e-5	1.0766	1.2739e-7	2.0099

Example 2. The orders of the fractional derivatives are $\alpha = 0.5,\ \beta = 0.5$. Here $2 + \alpha - 2\beta = 1.5$ and as it can be seen in Table 2, first order accuracy in time is obtained for the Grünwald-Letnikov approximation of the fractional derivatives. For the Lubich approximation and Method 1 the order of accuracy in time tends to 1.5 in the case of the compact scheme. For the usual second order approximation of the Laplacian we had not enough memory to compute the solution on grids that are sufficiently refined in both space directions and in time. Thus, we can not show the 1.5 order of accuracy here, but the initial results are similar to those in the case of the compact scheme, so we may expect similar behaviour on finer (in space and time) grids. For Method 2 the convergence is in accordance with the order of approximation in time – first order for the Grünwald-Letnikov and second for the Lubich approximation. Method 3 is not stable in most of the cases.

We also used the explicit method, and as in the previous example, about 50000 time steps had to be used in order to ensure stability for $N_x = N_y = 100$.

Example 3. The orders of the fractional derivatives are $\alpha = 0.9,\ \beta = 0.9$. In this case $2 + \alpha - 2\beta = 1.1$ and it is seen in Table 3 that first order accuracy in time is obtained for the Grünwald-Letnikov approximation of the fractional derivatives. For the Lubich approximation and Method 1 the order of accuracy in time tends to 1.1 in the case of the compact scheme. For the usual second order approximation of the Laplacian it can be almost seen that the order of convergence tends to 1.1. Unfortunately we have not enough memory to compute the solution on finer grids. For Method 2 the convergence is in accordance with the order of approximation in time – first order for the Grünwald-Letnikov and second for the Lubich approximation. Here Method 3 is stable and the orders of convergence are the same as for Method 2. It may be noticed that the order of convergence on the last line (compact scheme, $N_t = 6400$) is worse than for $N_t = 3200$. This is due to the fact that $N_x = 25$ points are not sufficient here, *i.e.*, the grid should be refined in space as well.

The explicit method is again used here, and as in the previous examples, about 50000 time steps had to be used in order to ensure stability for $N_x = N_y = 100$.

Example 4. The orders of the fractional derivatives are $\alpha = 0.9,\ \beta = 0.1$. Here $2 + \alpha - 2\beta = 2.7$ and Table 4 confirms that correspondingly first and second order accuracy in time is obtained for the Grünwald-Letnikov and

TABLE 7. The error $\delta(U)$ and the order l for $\alpha = 0.5,\ \beta = 0.9,\ 2+\alpha-2\beta = 0.7$

N_t	$N_x = N_y$	Method 1		Method 2		Method 3	
First order Grünwald-Letnikov approximation in time							
100	100	2.1472e-3		8.3602e-3		8.6590e-3	
200	100	8.0142e-4	1.4218	4.2534e-3	0.9749	4.3451e-3	0.9948
400	100	3.8416e-4	1.0609	2.1490e-3	0.9850	2.1773e-3	0.9968
800	100	5.0113e-4	stable	1.0822e-3	0.9897	1.0910e-3	0.9969
1600	100	4.4304e-4	0.1777	5.4450e-4	0.9910	5.4721e-4	0.9955
3200	100	3.3983e-4	0.3826	2.7436e-4	0.9889	2.7520e-4	0.9916
6400	100	2.4209e-4	0.4893	1.3891e-4	0.9819	1.3916e-4	0.9837
100	100	2.1472e-3		8.3602e-3		8.6590e-3	
200	200	8.0097e-4	1.4226	4.2514e-3	0.9756	4.3431e-3	0.9955
400	400	3.8691e-4	1.0498	2.1464e-3	0.9860	2.1747e-3	0.9979
800	800	5.0414e-4	stable	1.0793e-3	0.9918	1.0880e-3	0.9991
Second order Lubich approximation in time							
100	100	6.2613e-3		1.6712e-4		1.3899e-4	
200	200	4.2141e-3	0.5712	6.3000e-5	1.4075	3.5898e-5	1.9531
400	400	2.7456e-3	0.6181	2.2155e-5	1.5077	9.1539e-6	1.9714
800	800	1.7531e-3	0.6472	7.4867e-6	1.5652	2.3225e-6	1.9787
Compact approximation in space, second order Lubich approximation in time							
100	25	6.2559e-3		1.7000e-4		1.3554e-4	
200	25	4.2056e-3	0.5729	6.3684e-5	1.4165	3.5025e-5	1.9523
400	25	2.7384e-3	0.6190	2.2317e-5	1.5128	8.9306e-6	1.9716
800	25	1.7492e-3	0.6466	7.5270e-6	1.5680	2.2570e-6	1.9844
1600	25	1.1029e-3	0.6654	2.4788e-6	1.6024	5.6545e-7	1.9969
3200	25	6.8965e-4	0.6774	8.0463e-7	1.6232	1.3905e-7	2.0238
6400	25	4.2890e-4	0.6852	2.5988e-7	1.6305	3.1960e-8	2.1213

Lubich approximation of the fractional derivatives. Where Method 3 is stable, it gives similar results to those, obtained by Method 2.

The explicit method is also used, and here about 400 time steps are enough to ensure stability for $N_x = N_y = 100$. Thus the case when α is much greater than β seems to be easier from the numerical point of view.

Example 5. The orders of the fractional derivatives are $\alpha = 0.1,\ \beta = 0.9$. Here $2+\alpha-2\beta = 0.3$ and the results in Table 5 show that the order of convergence for Method 1 likely tends to 0.3, but very slowly. For Method 2 there is first order of convergence using the Grünwald-Letnikov formulas and 1.3 order of convergence using the Lubich formulas. Second order convergence may be obtained only by Method 3 in combination with the Lubich approximation, but according to the last row in the table, it may be not stable.

The numerical experiments show that the explicit method in this case is unstable even for $N_t = 2\times10^6$, $N_x = N_y = 25$.

Example 6. The orders of the fractional derivatives are $\alpha = 0.1,\ \beta = 0.5$. In this example $2+\alpha-2\beta = 1.1$ and it can be seen in Table 6 that first order of convergence is obtained for the Grünwald-Letnikov formulas. Method 1 in combination with the Lubich formulas shows 1.1 order of convergence, and Method 2 shows second order convergence. Method 3 is not stable in most of the cases.

As in the previous example, the explicit method is again unstable even for $N_t = 2\times10^6$, $N_x = N_y = 25$.

Example 7. The orders of the fractional derivatives are $\alpha = 0.5,\ \beta = 0.9$. Here $2+\alpha-2\beta = 0.7$ and in Table 7 we may observe 0.7 order of convergence for Method 1, first order convergence for Method 2 in combination with the Grünwald-Letnikov approximation, 1.7 order of convergence for Method 2 in combination with the Lubich approximation and correspondingly first and second order convergence for Method 3 using the Grünwald-Letnikov or Lubich formulas. We may notice that for the Grünwald-Letnikov approximation and Method 1 the order of convergence is not established in the first 4 rows of the table, and for $N_t = 800$ the error is greater than for $N_t = 400$. However the method is stable, and for larger values of N_t the order of convergence tends to 0.7.

The explicit method is again unstable even for $N_t = 2 \times 10^6$, $N_x = N_y = 25$.

CONCLUSIONS

The numerical experiments demonstrate that the orders of convergence of the proposed numerical schemes are in accordance with their orders of approximation. The first and the second method for choosing of the additional term in the ADI schemes appear to be unconditionally stable. Although the third method is not unconditionally stable, it appears to work well in cases, where the first two methods destroy the second order (Lubich) approximation of the equation in time.

The most time-consuming part of the numerical method is the computation of the finite time-fractional derivatives. In order to compute them, several gigabytes of memory are needed to store the solution U_{ij}^k, $i = 0, \ldots, N_x$, $j = 0, \ldots, N_y$, $k = 0, \ldots, N_t$, on relatively fine grids (1000×1000 grid points and for 1000 time steps). Double this memory would be needed if the time for computations has to be optimized – in such cases not only the solution, but its discrete Laplacian ΛU_{ij}^k should also be stored at all grid points and on all time levels. For the compact scheme instead of ΛU_{ij}^k we should keep $\Theta_x \Theta_y U_{ij}^k$ and ΘU_{ij}^k at all grid points and on all time levels.

With the compact scheme we may achieve the same accuracy as with the usual second order approximation of the Laplacian, but using much less grid points. This will allow us to save a lot of memory and to compute the solution much faster. Of course, the compact scheme may be used only for sufficiently smooth solutions.

The future work includes:

- Theoretical analysis of stability and convergence;
- Parallelization of the algorithms, which will result in much faster computations;
- Development of iterative solvers for the 2D implicit schemes instead of using ADI methods. This would allow us to avoid the restrictions imposed by the order of approximation of the additional term in Methods 1 and 2 and the restrictions for stability in Method 3;
- Development of techniques for truncation of the “tail” in the finite time-fractional derivatives formulas, which will result in faster computations and less memory use;
- Numerical experiments for practical problems with applications in industry.

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